概率程序的代数程序分析

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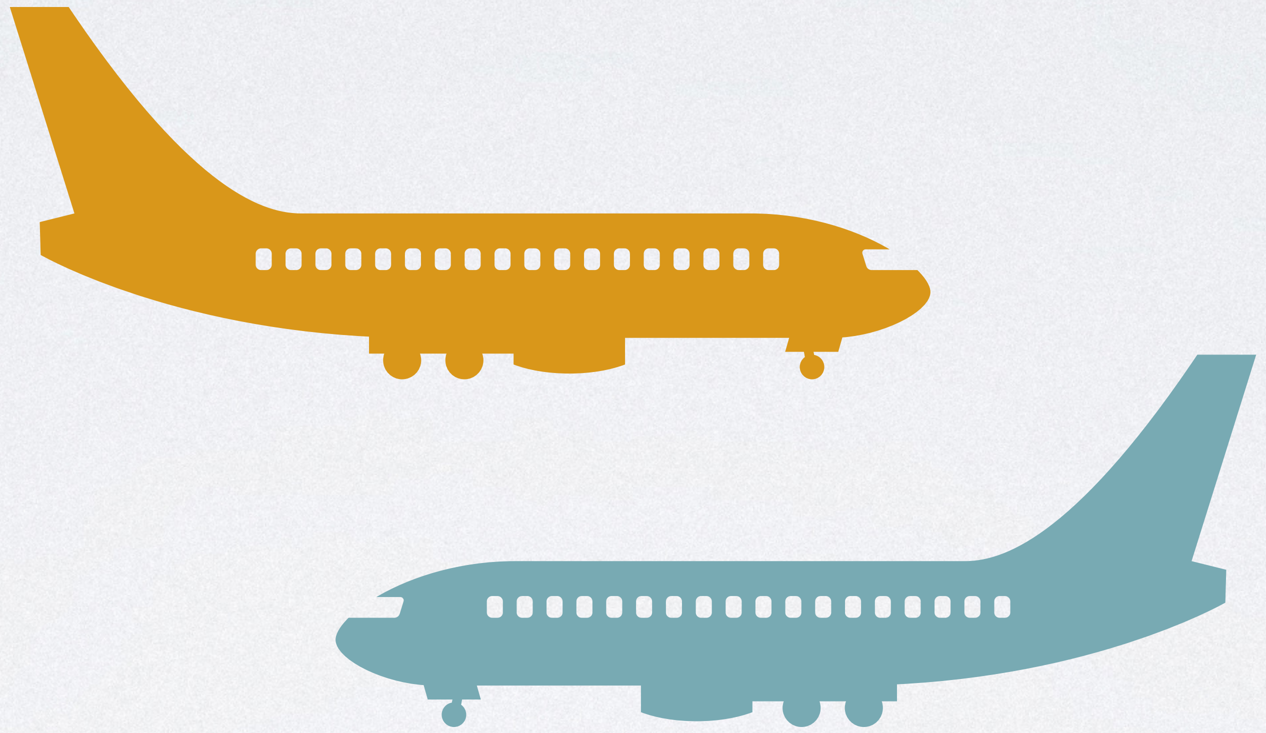
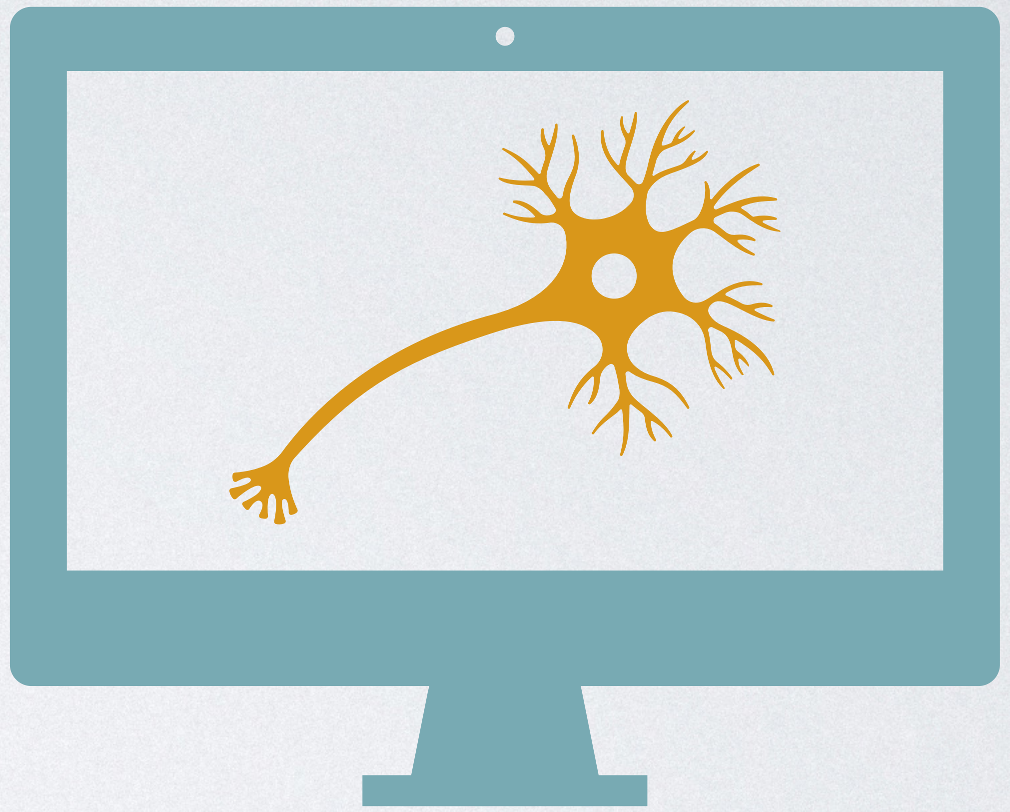
与 Jan Hoffmann 和 Thomas Reps 的合作工作

Probabilistic Systems are Becoming Pervasive

Randomized Algorithms
(improve efficiency)



Cyber-Physical Systems
(model uncertainty)



Artificial Intelligence
(describe statistical models)

Probabilistic Programs

Draw random **data** from distributions

Change **control-flow** at random

Image sources: https://www.libertystorch.info/2022/02/21/the-grab-bag/; https://www.stockvault.net/photo/179872/at-a-crossroads-decisions-and-choices-concept-with-large-arrow-signs.

Probabilistic Programs

- True randomness
- A distribution on execution paths
- Probabilistic nondeterminism

```
if
| prob(1/3) → choice ≔ 1
| prob(1/3) → choice ≔ 2
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choice :∈p (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3)
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Demonic Programs

- Dijkstra's **Guarded Command Language** (GCL)
- A set of execution paths
- Demonic nondeterminism

```
if
| true → prize ≔ 1
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prize $:\in_d$ {1,2,3}

Example: Monty Hall

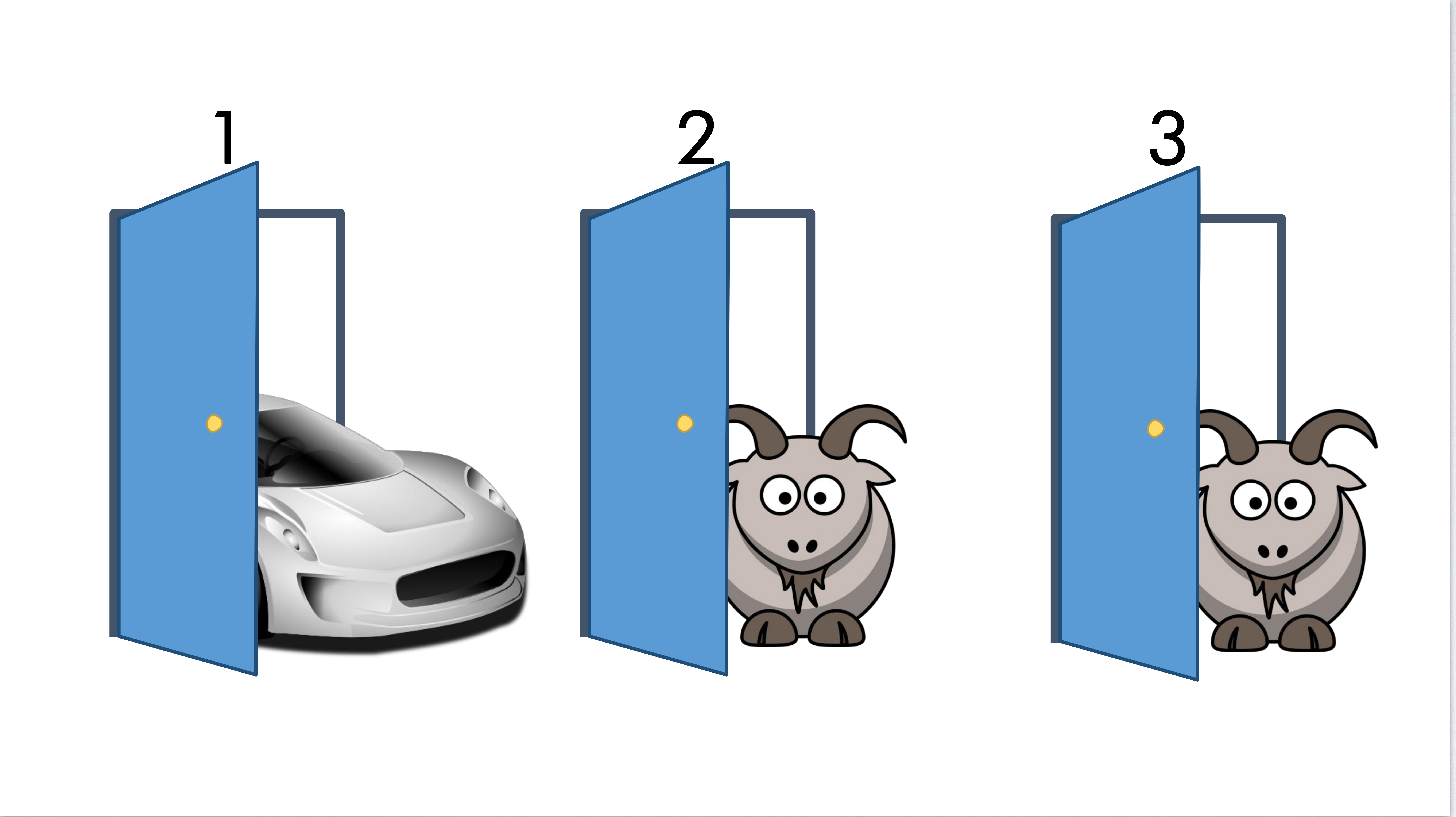

Image source: Maria Gorinova's Advances in Programming Languages (Guest Lecture) slides on Probabilistic Programming.

Example: Monty Hall

```
prize :∈d {1,2,3};
choice :∈p (1 @ 1/3 | 2 @ 1/3 | 3 @ 1/3);
host :∈d {1,2,3} \ {prize,choice};
if switch then
  choice :∈d {1,2,3} \ {choice,host}
fi
```

Example: Monty Hall

- McIver and Morgan's **probabilistic Guarded Command Language** (pGCL)
- Combine two forms of nondeterminism:
  - Probabilistic
  - Demonic

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$\mathbb{P}(choice = prize) = ?$

Example: Monty Hall

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- Combine two forms of nondeterminism:
  - Probabilistic
  - Demonic
- "Demons" minimize the probability

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Example: Failure Modeling

```
fail := FALSE;
c :∈d {0,1,2};
while not(fail) and c > 0 do
  fail :∈p (TRUE @ 0.1 | FALSE @ 0.9 );
  c := c - 1
od
```

Example: Failure Modeling

- An example of **probabilistic modeling checking**
- Send c messages, each with a failure probability 0.1

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fail := FALSE;
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Example: Failure Modeling

- An example of **probabilistic modeling checking**
- Send c messages, each with a failure probability 0.1
- What is the probability of success?

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fail := FALSE;
c :∈d {0,1,2};
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  c := c - 1
od
```

$\mathbb{P}(\mathit{fail} = \mathit{FALSE}) = ?$

Example: Abstraction

```
fail := FALSE;
[c=0] :∈d {TRUE,FALSE};
while not(fail) and not([c=0]) do
  fail :∈p (TRUE @ 0.1 | FALSE @ 0.9 );
  [c=0] :∈a {TRUE,FALSE}
od;
```

Example: Abstraction

- Program analysis introduces **abstraction**
- **Predicate Abstraction**
  - [c=0] is a Boolean variable

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Example: Abstraction

- Program analysis introduces **abstraction**
- **Predicate Abstraction**
  - [c=0] is a Boolean variable
- Abstraction nondeterminism
  - Maximize —> Upper bound
  - Minimize —> Lower bound

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while not(fail) and not([c=0]) do
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How to automate such quantitative reasoning about probabilistic programs?

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Examples

What is the probability that an assertion holds?

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What is the expected value of an expression?

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Examples

What is the probability that an assertion holds?

What is the expected value of an expression?

What is the expected time complexity of a program?

Challenge I:
How to support multiple confluence operations?

… $:\in_p$ …

… $:\in_d$ …

… $:\in_a$ …

Semantic Algebras

- **Kleene Algebras**: A **compositional** and **flexible** framework for program semantics

Program Construct	Algebraic Representation
A program S	An interpretation $\mathbb{S}$ of S into the algebra
Branching between A and B	$\mathbb{A} \oplus \mathbb{B}$
Sequencing of A and B	$\mathbb{A} \otimes \mathbb{B}$
Iteration (i.e., loop) of A	$\mathbb{A}^*$
"`abort`", "`skip`"	$\underline{0}, \underline{1}$

Do Kleene Algebras Suffice?

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```
if
| true → x := 1
| true → x := 2
| true → x := 3
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$$([\mathbf{true}] \otimes x \coloneqq 1)$$
$$\oplus ([\mathbf{true}] \otimes x \coloneqq 2)$$
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Do Kleene Algebras Suffice?

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if
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```

$$
\begin{aligned}
&([\mathbf{true}] \otimes x \coloneqq 1) \\
\oplus\ &([\mathbf{true}] \otimes x \coloneqq 2) \\
\oplus\ &([\mathbf{true}] \otimes x \coloneqq 3)
\end{aligned}
$$

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Probabilities sum up to 2!

Our Approach: Markov Algebras

- Key observation: Probabilistic programs have **multiple confluence operations**

$$\left\langle M, \sqsubseteq, \otimes, {}_{\phi}\oplus, \sqcap, \underline{0}, \underline{1} \right\rangle$$

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Program denotations form a CPO

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Sequencing, branching, and nondeterministic-choice

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Easy to extend with more confluence operations!

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Sequencing, branching, and nondeterministic-choice

$\underline{0}$ interprets **`abort`**
$\underline{1}$ interprets **`skip`**

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Easy to extend with more confluence operations!

$$(a \otimes b) \otimes c = a \otimes (b \otimes c)$$

$$a \otimes \underline{1} = \underline{1} \otimes a = a$$

$$a\, {}_{\phi}\oplus\, b = b\, {}_{\overline{\phi}}\oplus\, a$$

$$a \sqcap a = a$$

...

Markov Algebras Suffice!

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```
while x>0 do
  x :∈p (x+1 @ 1/2 | x-1 @ 1/2)
od
```

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$$\mu S \,.\, ((x := x{+}1 \,{}_{1/2}\!\oplus\, x := x{-}1) \otimes S) \,{}_{[x>0]}\!\oplus\, \mathbf{skip}$$

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Recursive Program Scheme

Challenge II:
How to carry out quantitative analyses efficiently?

```
while prob(2/3) do
  x ≔ x + 1
od
```

Iterative Program Analysis

```
while prob(2/3) do
  x := x + 1
od
```

$$\mu S\,.\,((\mathtt{x} := \mathtt{x{+}1}) \otimes S)_{[2/3]}\oplus \textbf{skip}$$

Iterative Program Analysis

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while prob(2/3) do
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$$\mu S\,.\,((\mathtt{x := x+1}) \otimes S)_{[2/3]}\oplus \mathbf{skip}$$

- Markov algebra for computing $\mathbb{E}[\Delta x]$
  - Sequencing: $r \otimes t \triangleq r + t$
  - Branching: $r_p\oplus t \triangleq p * r + (1 - p) * t$

Iterative Program Analysis

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while prob(2/3) do
  x := x + 1
od
```

$\mu S\,.\,((\mathrm{x} := \mathrm{x}{+}1) \otimes S)_{[2/3]}\oplus\ \mathbf{skip}$

$\kappa^{(0)} = 0$

$\kappa^{(1)} = 2/3 * (1 + \kappa^{(0)}) + 1/3 * 0 = 2/3$

$\kappa^{(2)} = 2/3 * (1 + \kappa^{(1)}) + 1/3 * 0 = 10/9$

$\cdots$

$\kappa^{(\infty)} = 2$

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$$\cdots$$
$$\kappa^{(\infty)} = 2$$

Need ∞ iterations to converge!

Non-iterative Program Analysis

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while prob(2/3) do
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  - Sequencing: $r \otimes t \triangleq r + t$
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Equivalent to solve:

$$s = 2/3 * (1 + s) + 1/3 * 0,$$

Analytical solution:

$$s = 2$$

No need for iteration!

Beyond Loops

```
proc X begin
  if prob(1/3) then
    skip
  else
    call X;
    call X
  fi
end
```

Beyond Loops

$$X = \mathbf{skip}_{1/3} \oplus (X \otimes X)$$

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⬇ Computing $\mathbb{P}[\text{terminate}]$

$$p = 1/3 * 1 + 2/3 * (p * p)$$

Beyond Loops

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Computing $\mathbb{P}[\text{terminate}]$

Non-linear!

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Newtons's method

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$$f(x) = 1/3 * 1 + 2/3 * (x * x)$$

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Newtons's method

$$f(x) = 1/3 * 1 + 2/3 * (x * x)$$

$$\Delta^{(i)} = (f(p^{(i)}) - p^{(i)}) + f'(p^{(i)}) * \Delta^{(i)}$$

$$p^{(i+1)} \leftarrow p^{(i)} + \Delta^{(i)}$$

Beyond Loops

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Newtons's method

$$f(x) = 1/3 * 1 + 2/3 * (x * x)$$

Linear!

$$\Delta^{(i)} = (f(p^{(i)}) - p^{(i)}) + f'(p^{(i)}) * \Delta^{(i)}$$

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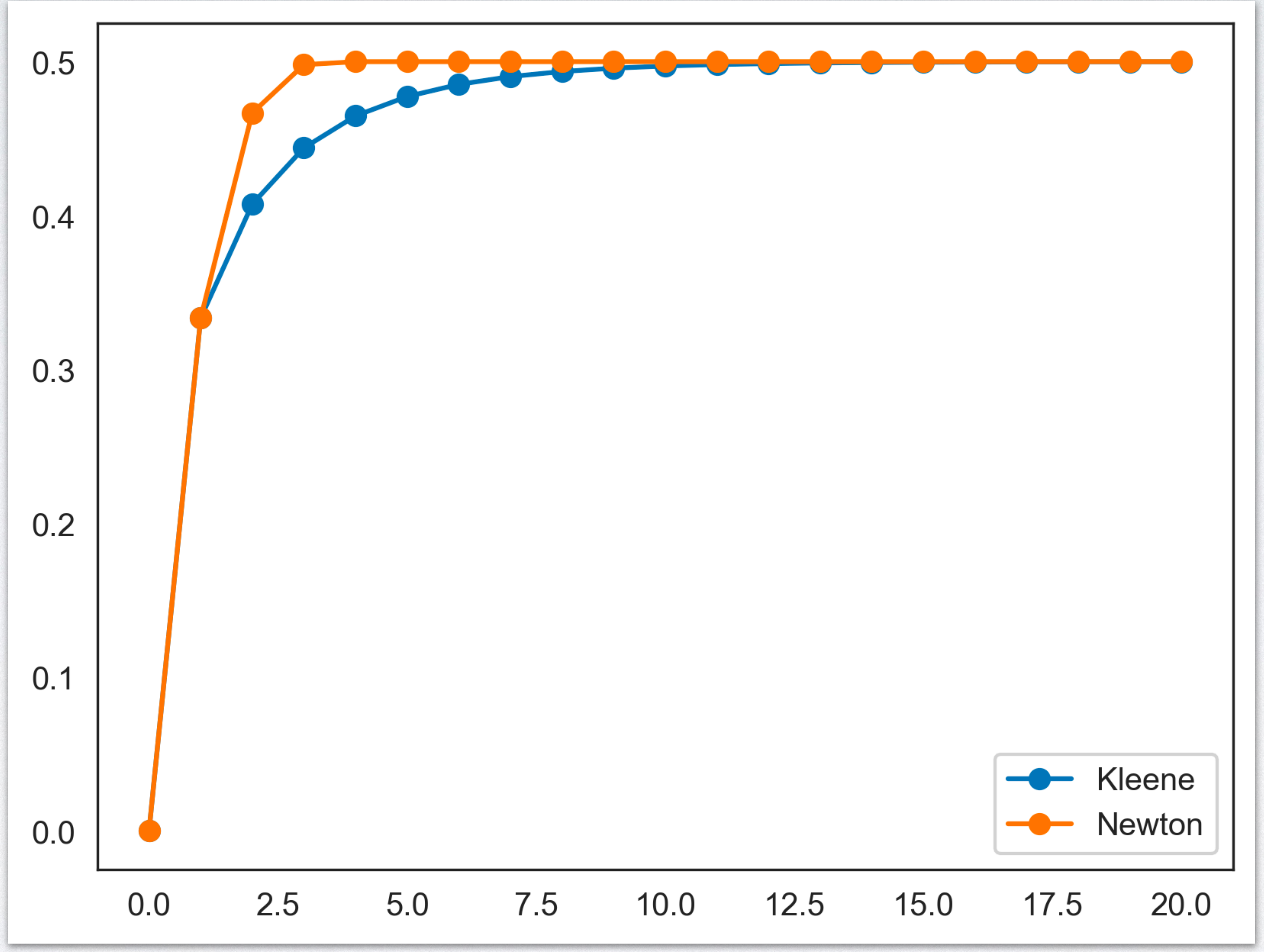
Newton's Method Converges Faster



Newtonian Program Analysis (NPA)

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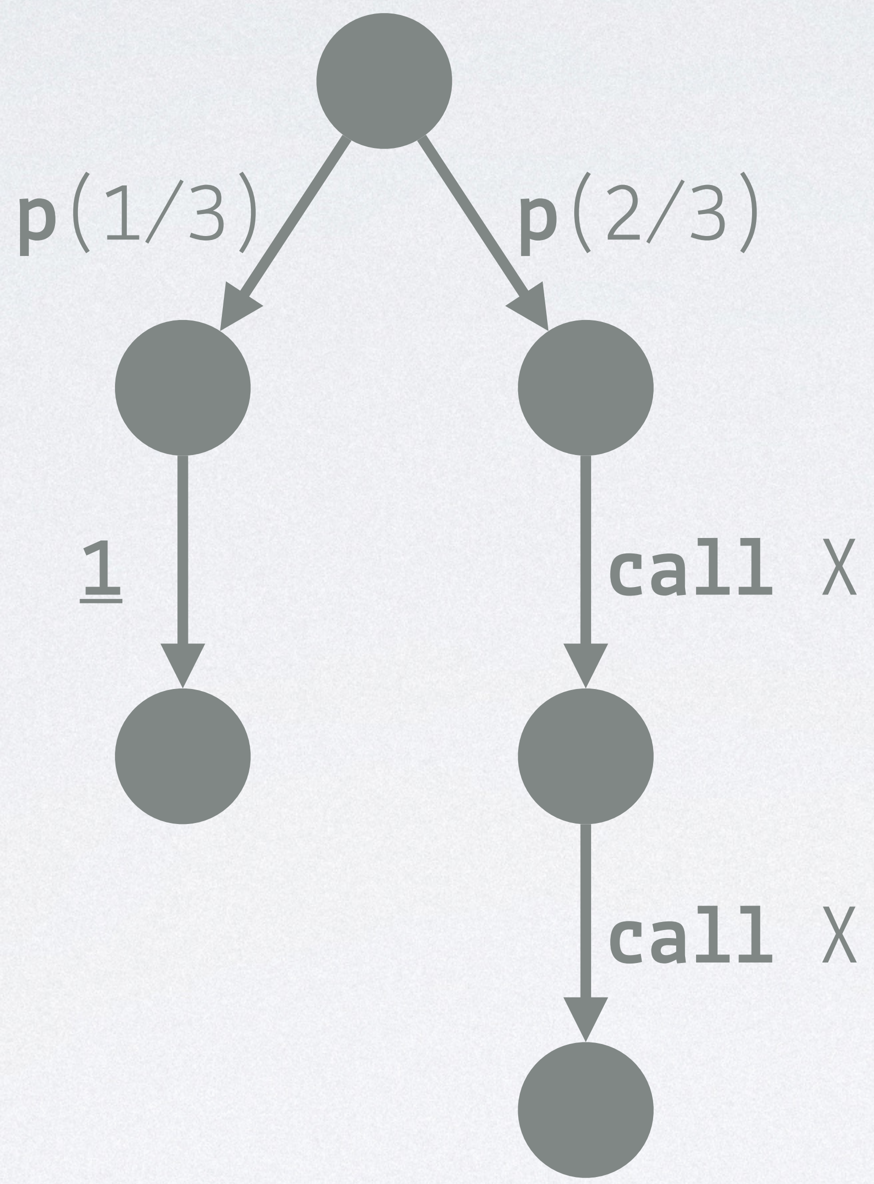

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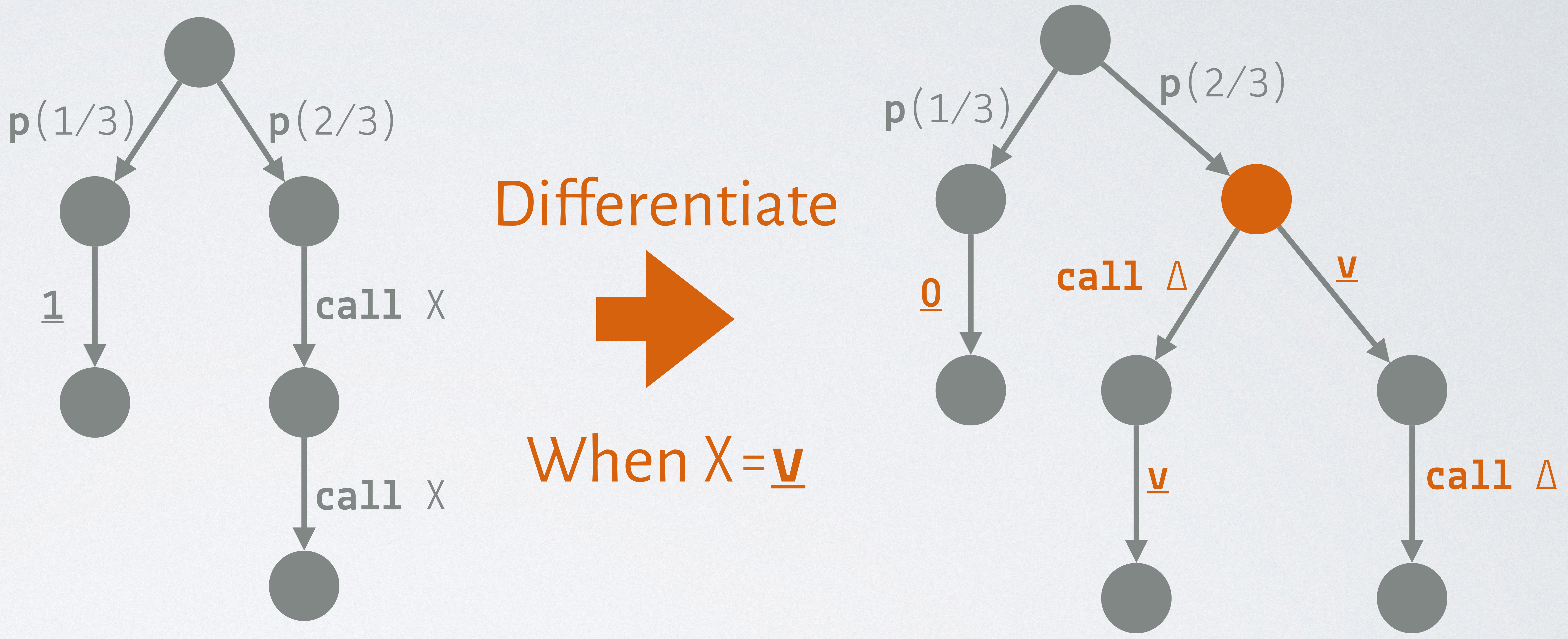

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  if prob(1/3) then
    skip
  else
    call X;
    call X
  fi
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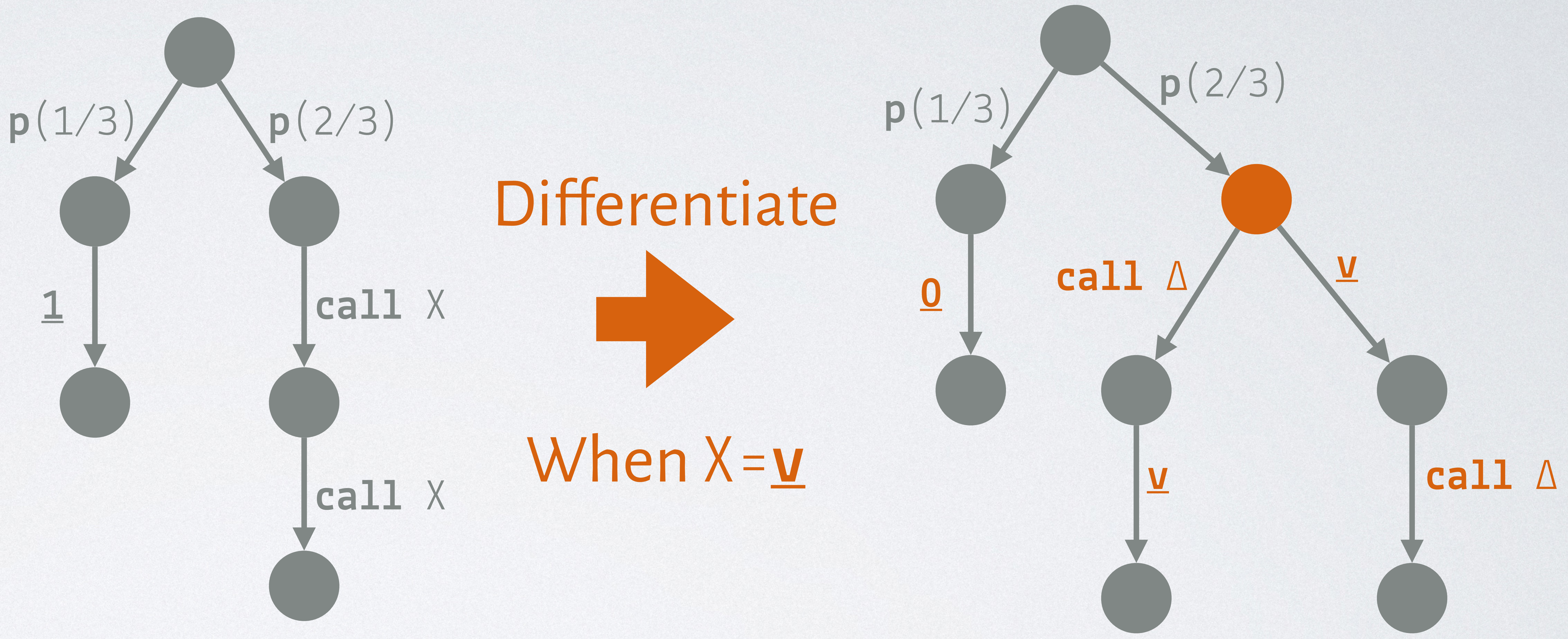
```



Every root-to-leaf path contains **at most one call**!

Esparza, J., Kiefer, S, and Luttenberger, M. 2010. Newtonian program analysis. J. ACM 57, 6, Article 33, (October 2010), 47 pages.

Newtonian Program Analysis (NPA)

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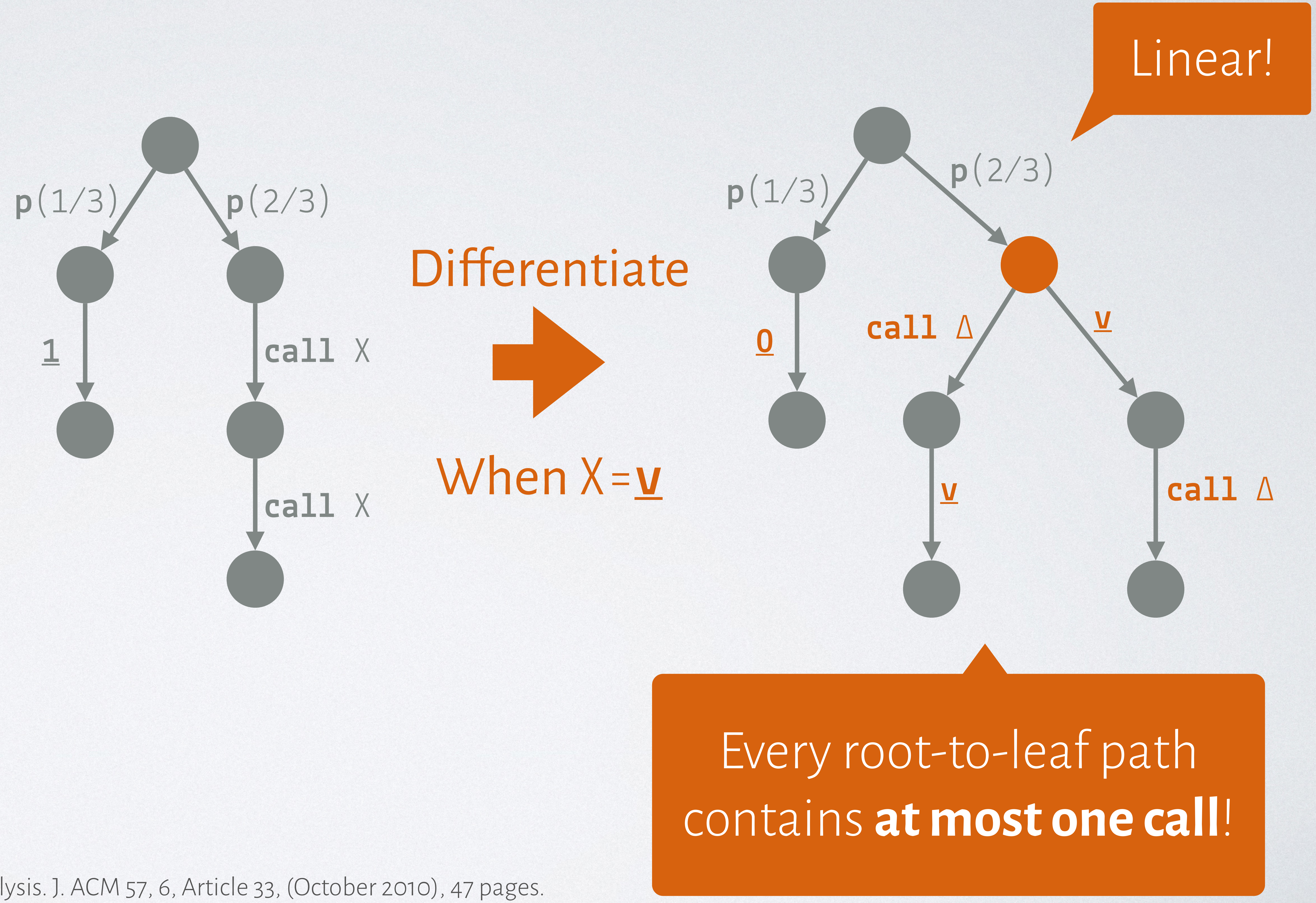

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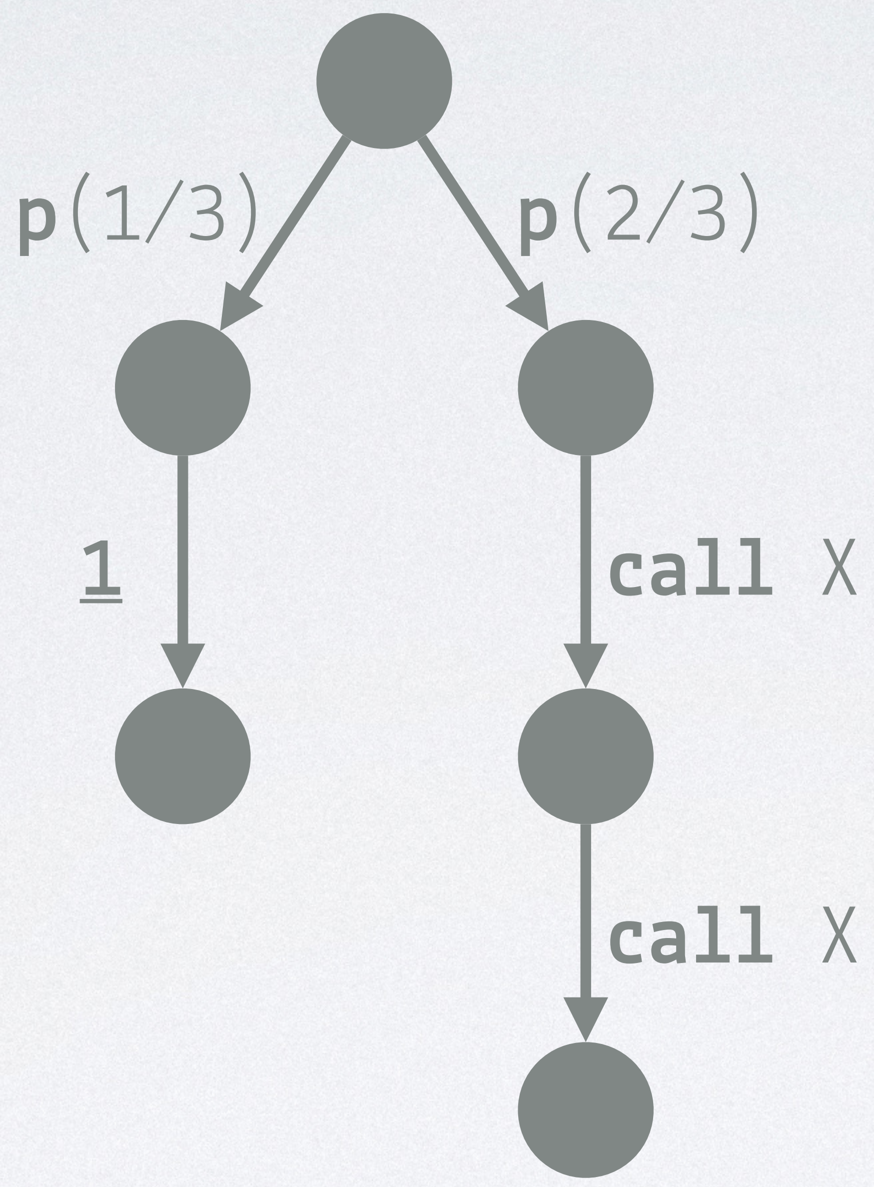
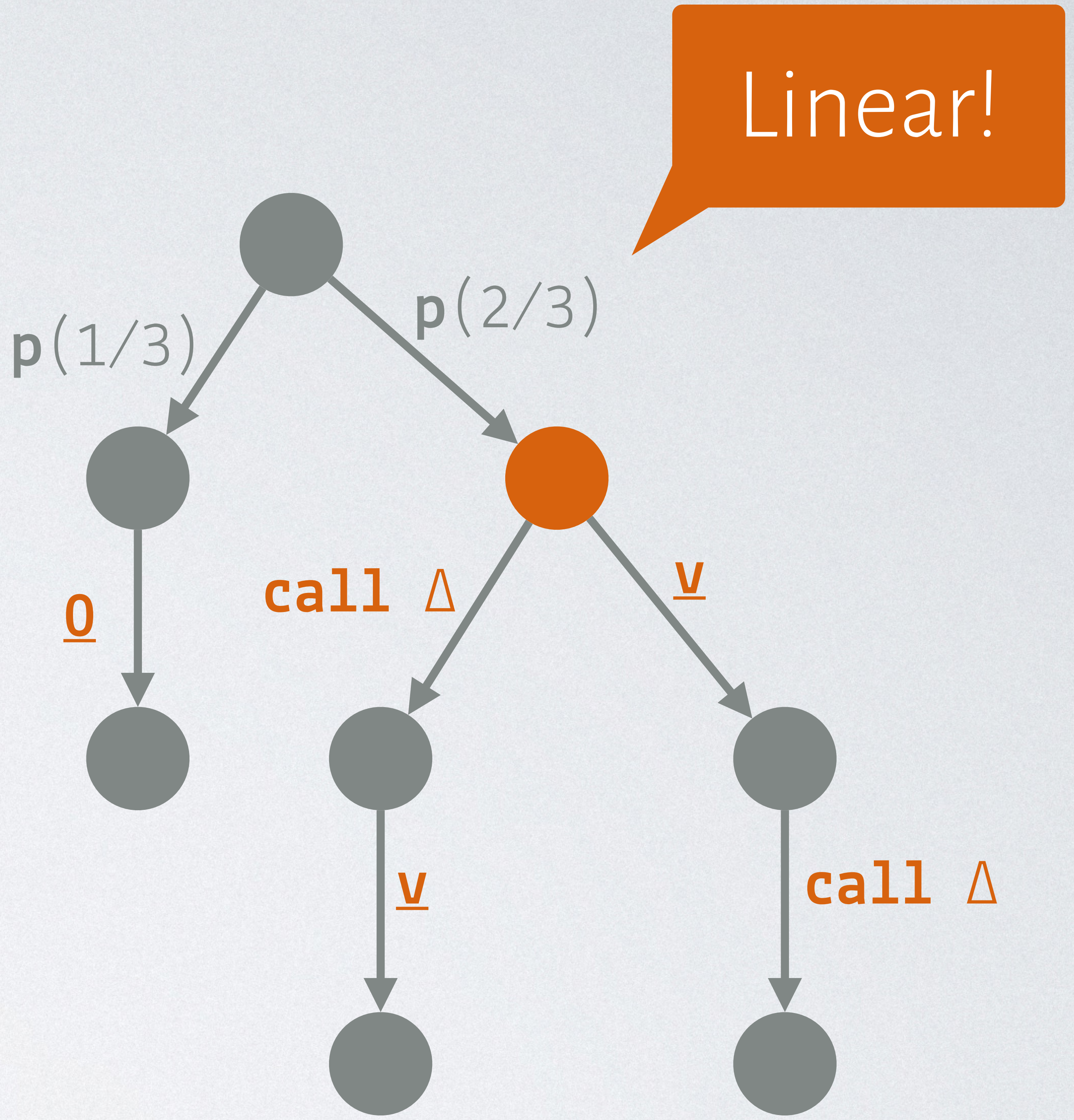


Differentiate

When X = **v**



$$\frac{d(f*g)}{dx} = \frac{df}{dx} * g + f * \frac{dg}{dx}$$

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Our Approach: NPA for pre-Markov Algebras

- Key idea: Apply Newton's method to **pre-Markov algebras**

- We develop a differentiation routine for **recursive program schemes**

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Support multiple confluences, loops, and recursion

- We develop a differentiation routine for **recursive program schemes**

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$$\langle M, \oplus, \otimes, {}_{\phi}\oplus, \sqcap, \underline{0}, \underline{1} \rangle$$

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$\oplus$ defines a partial order and gives an additive structure

$$\langle M, \oplus, \otimes, {}_{\phi}\oplus, \sqcap, \underline{0}, \underline{1} \rangle$$

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Equation System

$$\langle M, \oplus, \otimes, {}_{\phi}\oplus, \sqcap, \underline{0}, \underline{1} \rangle$$

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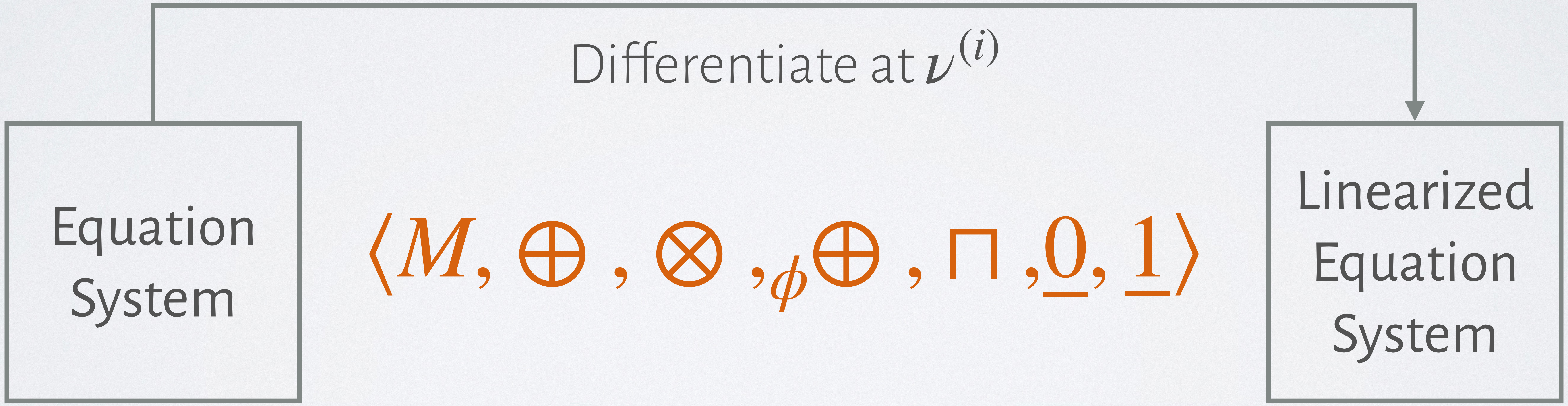

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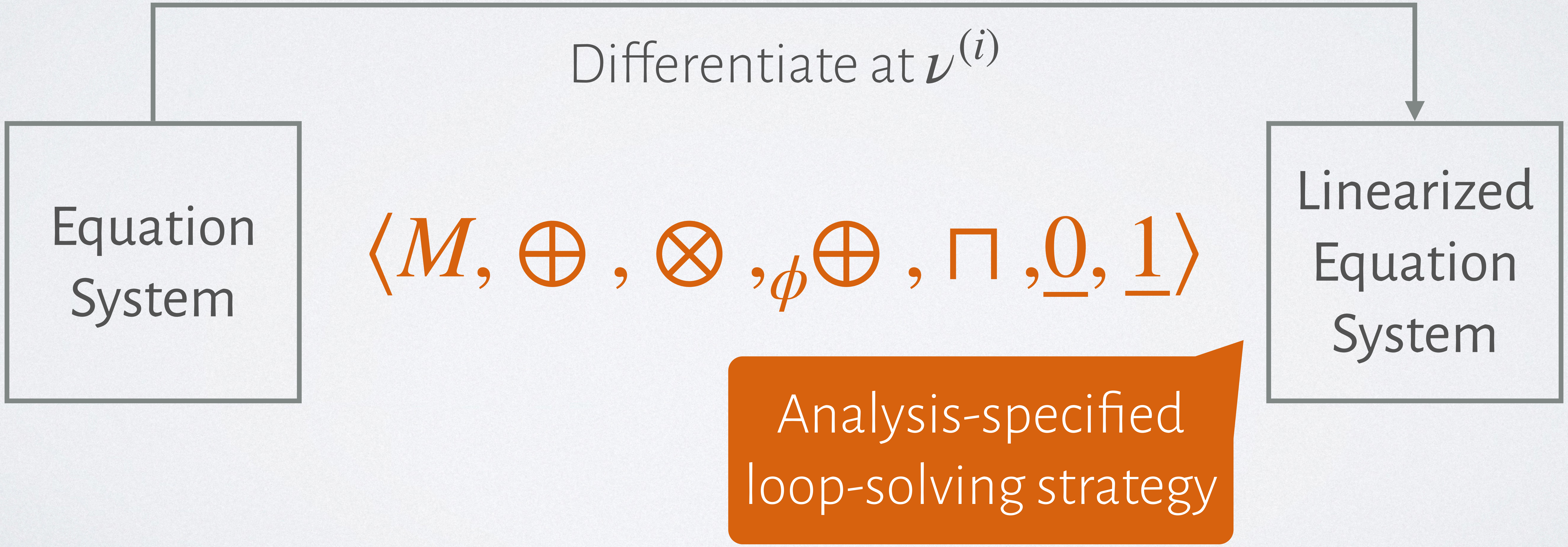

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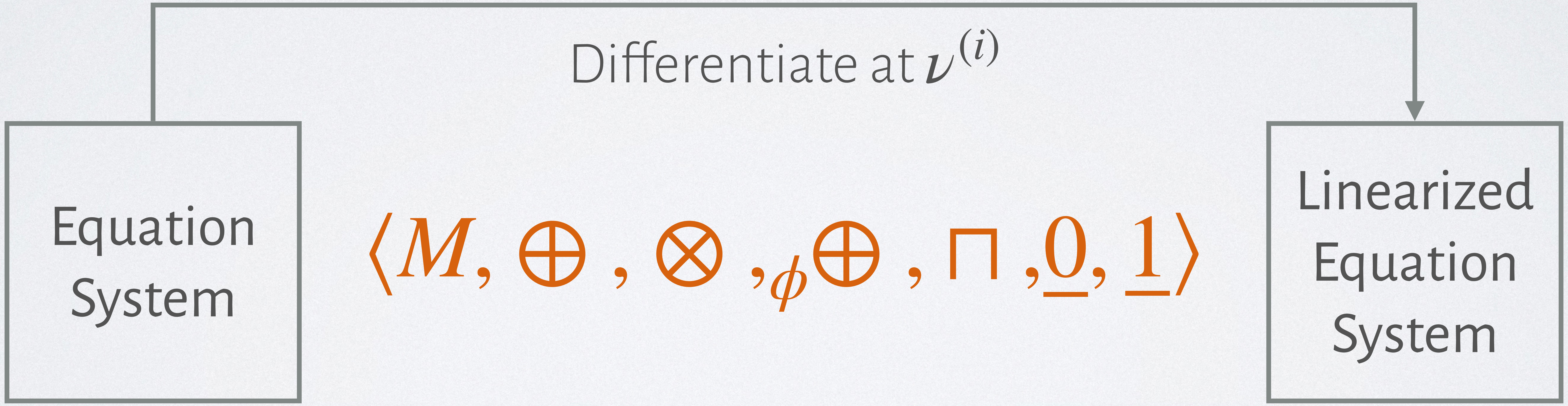

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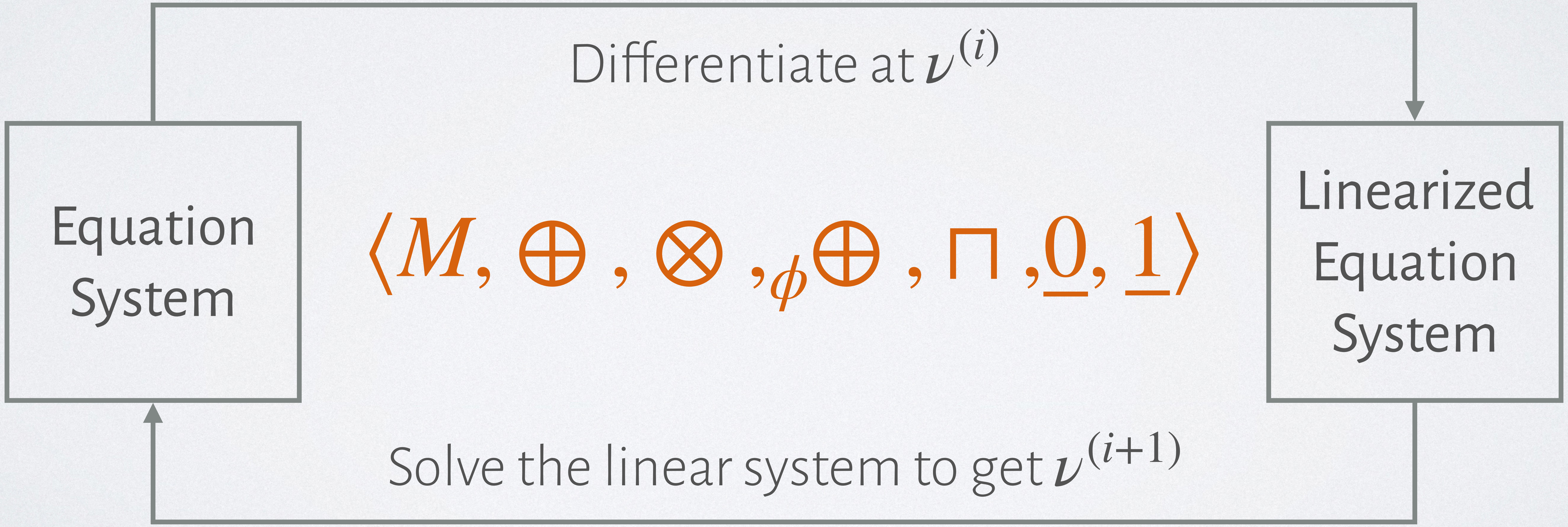
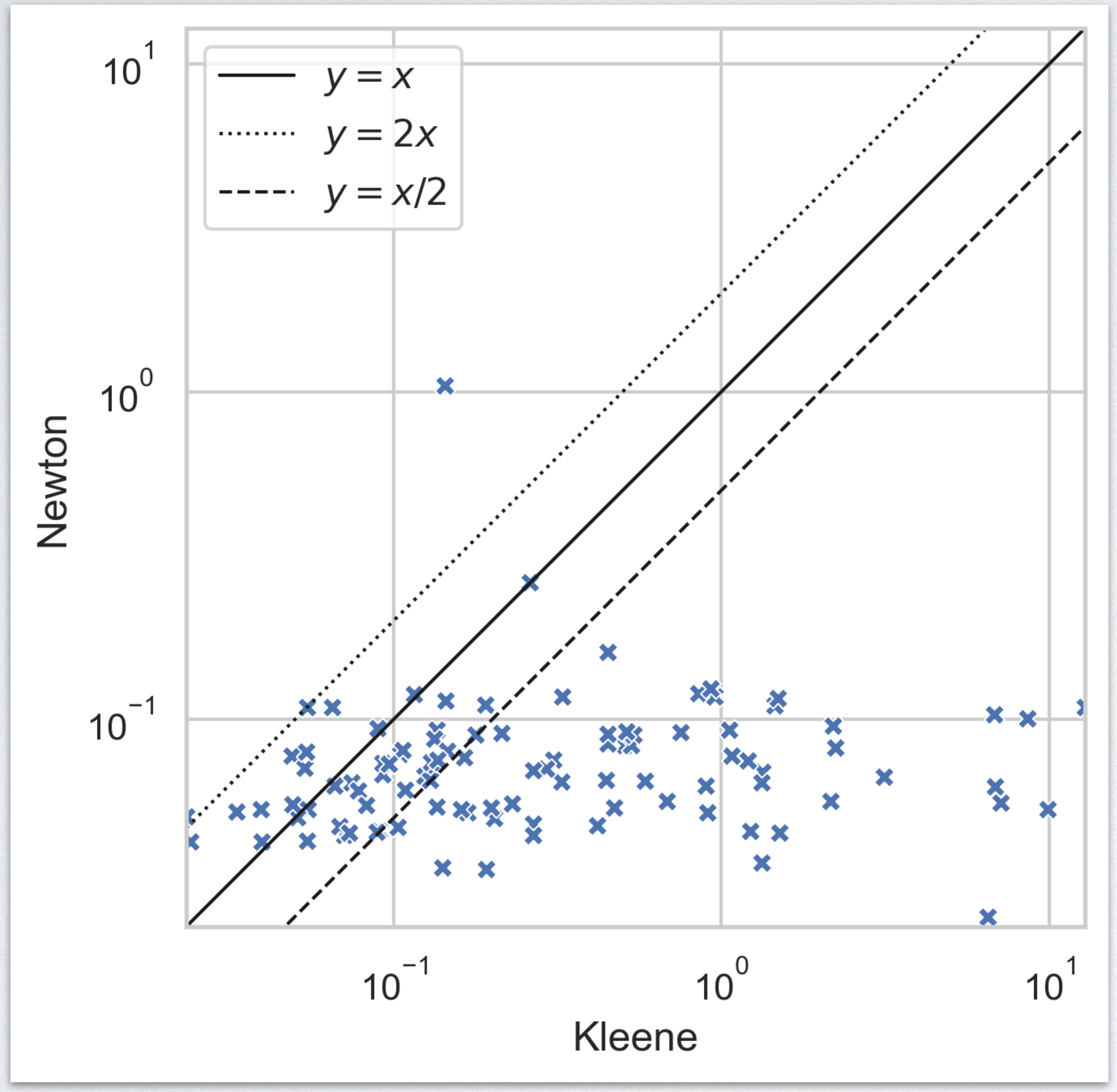
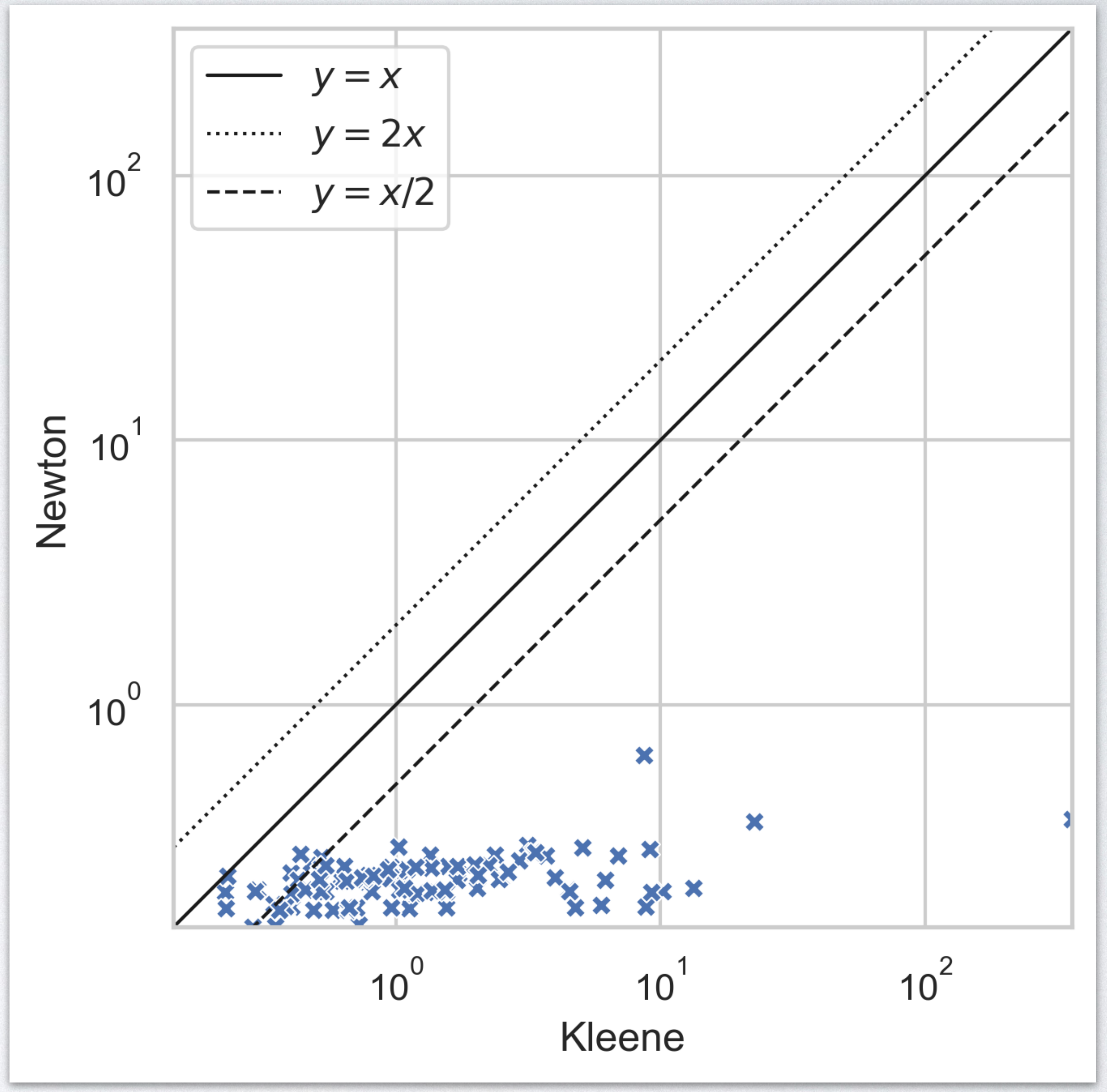


Preliminary Evaluation



Reaching Probability



Expected Reward

Our Papers

- Di Wang, Jan Hoffmann, Thomas Reps (2018). **PMAF: An Algebraic Framework for Static Analysis of Probabilistic Programs**. In *PLDI'18*.
- Di Wang, Jan Hoffmann, Thomas Reps (2019). **A Denotational Semantics for Low-Level Probabilistic Programs with Nondeterminism**. In *MFPS'19*.
- Di Wang, Thomas Reps (2023). **Newtonian Program Analysis of Probabilistic Programs**. *Working Paper*.

Towards a flexible and efficient framework for program analysis of probabilistic programs

- ☑ **Semantics:** Markov Algebras for Multiple Kinds of Confluences
- ☑ **Algorithm:** Newton's Method for pre-Markov Algebras

Towards a flexible and efficient framework for program analysis of probabilistic programs

- ☑ **Semantics:** Markov Algebras for Multiple Kinds of Confluences
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- ☑ **Representation:** Construction of Recursive Program Schemes